Earnings dynamics and top-earnings inequality*

Karl Harmenberg[†] Raysa Lizarraga[‡]

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We introduce a simple generalization of the canonical permanent-transitory earnings process, a square-root process. The square-root process generates a Pareto tail in earnings and is able to match the dynamics of top-earnings inequality over the life cycle while retaining a good match to the covariance structure of earnings. By contrast, the canonical model fails to match the dynamics of top-earnings inequality over the life cycle. Since our square-root process is simple, with only one state variable, it can easily be used in structural macroeconomic models.

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[†]University of Oslo and BI Norwegian Business School, karl.harmenberg@econ.uio.no.
[‡]BI Norwegian Business School, raysa.lizarraga@bi.no.

1. Introduction

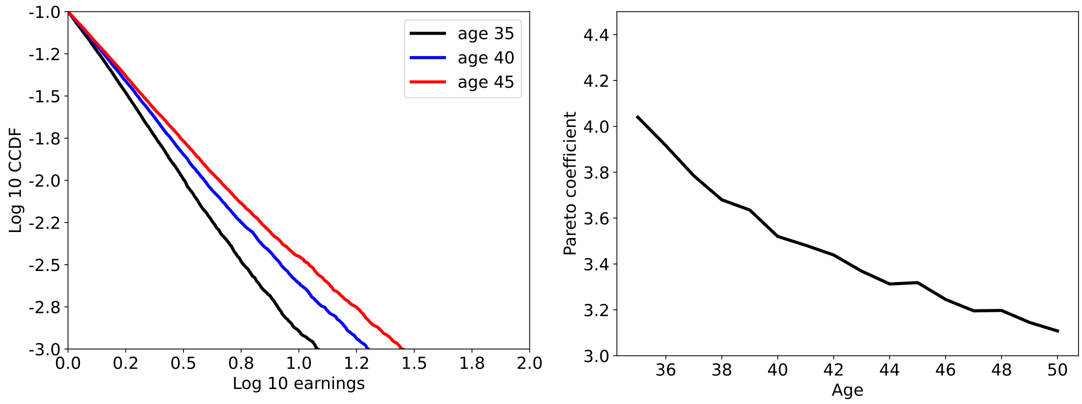
The canonical model of earnings dynamics, which is the backbone of quantitative macroeconomics, is an earnings process composed of a permanent component and a transitory component, where the permanent component of earnings follows a highly persistent $AR(1)$ process.[1] This canonical earnings process provides a good match to the covariance structure of earnings and, since it only features a single state variable, is easy to use in consumption-savings models. Further, the stationary distribution of an $AR(1)$ process is a normal distribution, and since a log-normal distribution provides a reasonable approximation of the earnings distribution, the canonical earnings process also provides an earnings distribution qualitatively consistent with the data.

Although the canonical model has been a tremendous success along all these dimensions, it provides a poor fit to top-earnings inequality. The central stylized fact of top-earnings inequality is that the top approximately follows a Pareto distribution (Pareto, 1896), i.e., the earnings distribution is "fractal" (Jones, 2015) at the top.[2] For example, the percentage gap in earnings between the top ten percent and the top one percent is the same as the gap between the top one percent and the top 0.1 percent. In Figure 1a, we illustrate this fact by showing that the Norwegian earnings distribution features a Pareto tail. An exact Pareto distribution corresponds to a straight line and, indeed, for each selected age, the data is very well approximated by a straight line. This fact is in stark contrast to the stationary distribution of the canonical earnings process, which is a log-normal distribution with a thin upper tail.

Despite the importance of the top of the earnings distribution for not only income inequality but also wealth inequality and macroeconomic outcomes such as aggregate savings, the quantitative literature has not been able to provide a satisfactory treatment of earnings dynamics consistent with documented top-earnings inequality. Several approaches have been tried. Castañeda et al. (2003) account for the earnings distribution by departing from the canonical model, and instead calibrate a Markov state to directly target the Lorenz curve of earnings. The Markov chain cannot, trivially since it only allows four states, generate a Pareto tail in earnings and is not calibrated to match facts on earnings dynamics. Hubmer et al. (2021) use the canonical model as their earnings process but, to be able to match top-earnings inequality, they adjust the earnings distribution ad hoc such that the top of the distribution

[1] Central references on the canonical model include Friedman and Kuznets (1945), Lillard and Willis (1978), MaCurdy (1982), Abowd and Card (1989), Storesletten et al. (2004), Meghir and Pistaferri (2004) and Low et al. (2010). Since the beginning of heterogeneous-agent macroeconomics (Imrohoroglu, 1989, Huggett, 1993, Aiyagari, 1994), the canonical model has been used as the key source of idiosyncratic risk.

[2] See De Vries and Toda (2022) for recent cross-country evidence.



(a) Top earnings are approximately Pareto distributed. **(b)** The age-specific Pareto coefficient is declining over the life cycle.

Figure 1: Both subfigures display Norwegian male earnings data. In (a), we show the top of the earnings distribution for different ages. The horizontal axis is log 10 earnings, normalized so that the threshold log earnings for belonging to the top ten percent is zero. The vertical axis is the log 10 of the complementary cumulative distribution function, such that -1 corresponds to the top ten percent, -2 corresponds to the top one percent, and so on. An exact Pareto distribution corresponds to a straight line. For privacy reasons, each data point displayed in the figure is the average of five earners. In (b), we plot the Pareto coefficient, i.e., the absolute values of the slopes from (a), from ages 35 to 50. For details, see Section 3.

follows a Pareto distribution with an exogenous Pareto coefficient. Neither approach generates top-earnings inequality as a result of an earnings process which is calibrated to be consistent with the dynamics of earnings at the micro level.

An extension of the canonical model which does generate a Pareto tail in earnings is a random walk for log earnings with a constant reset probability (interpreted as death or as "falling off the ladder").[3] Although this parsimonious model does generate a steady-state earnings distribution featuring a Pareto tail, it cannot account for the dynamics of top-earnings inequality. As shown by Gabaix et al. (2016), the dynamics of top-earnings inequality are extremely slow in this class of model, measured in hundreds of years, and the model is thus a non-starter for accounting for a rise in top-earnings inequality over the life cycle, or for the increase in top-earnings inequality seen in the US over the last fifty years. This failure of the canonical model is particularly salient over the life cycle. In Figure 1b, we show the age-specific Pareto coefficient for the top ten percent of our sample from Norway, over the life cycle. The

[3]Early contributions can be found in Champernowne (1953) and Simon (1955). More recent contributions include Nirei and Aoki (2016) and Toda and Walsh (2015). Beare and Toda (2022) provide a framework which nests this entire literature where earnings growth depends on a Markov chain with reset. The canonical model augmented with death/reset has been used in the quantitative-applied macroeconomics literature by McKay (2017), Carroll et al. (2017, 2020) and Harmenberg and Öberg (2021).

Pareto coefficient is declining substantially over the life cycle as top-earnings inequality increases.[4]

In this paper, we provide a simple generalization of the canonical model which does generate a Pareto tail in earnings, is able to account for the increase in top-earnings inequality over the life cycle, and retains the desirable properties of the canonical model. Concretely, we propose the following generalization of the canonical model for the dynamics of persistent earnings z_t:

$$z_t = \theta \bar{z} + (1-\theta) z_{t-1} + \left(\Sigma^2 + \Psi^2 z_{t-1}\right)^{1/2} \nu_t, \ \nu_t \sim N(0,1).$$

If $\Psi = 0$, we recover the canonical model, $z_t = \theta\bar{z} + (1-\theta)z_{t-1} + \Sigma\nu_t$. When $\Psi \neq 0$, earnings risk is state dependent, increasing with permanent earnings, and we obtain a *square-root process* for log permanent earnings. This process turns out to yield a Pareto tail in permanent earnings, not only in a steady state but also along a transition path. Further, the dynamics of top-earnings inequality can be quantitatively fast, matching the empirical facts.

Since our proposed earnings process is a strict generalization of the canonical model, it also retains the desirable properties of the canonical model, and we can run a horse race between the two models. The calibrated square-root process ($\Psi \neq 0$) successfully matches the dynamics of top-earnings inequality over the life cycle while the canonical model ($\Psi = 0$) simply cannot. Yet, the calibrated square-root process matches the covariance structure of earnings as well as the canonical model. Further, it also retains the parsimony of the canonical model, with only one state variable. We conclude that the square-root process would serve well as a new backbone of quantitative macroeconomics and household finance, especially for questions where top-earnings inequality is central.

Further related literature Our paper connects to the literature on power laws in economics and finance (see Gabaix (2009) and Benhabib and Bisin (2018) for excellent overviews). A main motivation of our paper is the view that power laws (e.g., the fact that earnings are Pareto distributed at the top) provide guidance for quantitative theories: our workhorse models ought to be consistent with the robust and stylized facts that power laws provide. This paper follows a tradition, cited in Footnote 3, where the Pareto tail in earnings emerges endogenously from a more primitive dynamic earnings process. An alternative and complementary approach is to derive an earnings distribution with a Pareto tail from a static economic theory with, e.g., assortative matching (Geerolf, 2017) or downward-sloping labor demand (Harmenberg, 2024).

[4] That the Pareto coefficient is declining over the life cycle has previously been documented by Badel et al. (2018) for the US, Canada, Denmark, and Sweden.

We also connect to a recent literature on earnings dynamics. In particular, we share with Guvenen et al. (2014), Arellano et al. (2017), and Guvenen et al. (2021) the view that the canonical model fails to match salient and robust stylized facts of earnings dynamics.[5] Whereas the focus of these papers is, in particular, on the higher-order moments of earnings shocks, we focus on the implied distribution of earnings generated by earnings processes. Guvenen et al. (2022) provide a tractable model which is able to match the higher-order moments of earnings shocks. Similarly, we provide in this paper a tractable model which matches the dynamics of top-earnings inequality over the life cycle.

More broadly, our earnings process serves as an input for answering macroeconomic questions for which top inequality matters. These include, but are not limited to, the savings behavior of the rich (Fagereng et al., 2019, Straub, 2019), the degree of partial insurance (Blundell et al., 2008, Kaplan and Violante, 2010), and optimal taxation at the top of the income distribution (Guner et al., 2016, Badel et al., 2020, Kindermann and Krueger, 2022).

Finally, on the technical side, we are not the first to employ a square-root process in economics. In particular, the Cox-Ingersoll-Ross model of short-term interest rates (Cox et al., 1985) is also a square-root process.

Structure of the rest of the paper In Section 2, we present the square-root earnings process and derive its theoretical properties. In Section 3, we calibrate the square-root earnings process and compare its properties with data and the canonical model. Section 4 concludes.

2. Theoretical model description

In this section, we introduce our earnings process and derive our theoretical results. Let log earnings y_t follow the process given by

$$dy_t = \theta(\bar{y} - y_t)dt + \left(\Sigma^2 + \Psi^2 y_t\right)^{1/2} dW_t \tag{1}$$

where W_t is a standard Brownian motion. This process is a strict generalization of an Ornstein-Uhlenbeck process, i.e., the continuous-time equivalent of an $AR(1)$. With $\Psi = 0$, we recover an Ornstein-Uhlenbeck process.

If $\Psi \neq 0$, we can without loss of generality assume that $\Sigma = 0$,[6] yielding a process on the following

[5] See Halvorsen et al. (2022) for documentation of higher-order stylized facts and Crawley et al. (2022) for a recent contribution to the literature on the canonical model, both using Norwegian administrative data.

[6] Introduce the change of variable $\hat{y}_t = y_t - \frac{\Sigma^2}{\Psi^2}$. Then Equation (1) can be rewritten as $d\hat{y}_t = \theta(\hat{\bar{y}} - \hat{y}_t)dt + \Psi\sqrt{\hat{y}_t}dW_t$

form:

$$dy_t = \theta(\bar{y} - y_t)dt + \Psi\sqrt{y_t}dW_t. \tag{2}$$

This is a *square-root process.* In contrast to an Ornstein-Uhlenbeck process, the square-root process implies that earnings risk is state-dependent: higher earnings are associated with higher earnings risk through the $\Psi\sqrt{y_t}$ factor multiplying the Brownian motion. It turns out that this slight tweak of the standard Ornstein-Uhlenbeck/$AR(1)$ process yields several appealing features, summarized by the following proposition.

Proposition 1. *Let log earnings y_t follow the square-root process given by*

$$dy_t = \theta\left(\bar{y} - y_t\right)dt + \Psi\sqrt{y_t}dW_t. \tag{3}$$

Assume that the initial distribution of log earnings is given by the gamma distribution,

$$f_0(y) = \frac{(\eta_0)^{-n}}{\Gamma(n)} y^{n-1}\exp(-y/\eta_0) \tag{4}$$

with $n = \frac{2\theta\bar{y}}{\Psi^2}$. Then the cross-sectional distribution of log earnings at a given point in time is also distributed according to a gamma distribution,

$$f_t(y) = \frac{(\eta_t)^{-n}}{\Gamma(n)} y^{n-1}\exp(-y/\eta_t) \tag{5}$$

where η_t evolves according to

$$d\eta_t = (\Psi^2/2 - \theta\eta_t)dt. \tag{6}$$

The distribution of log earnings at time t has variance given by $n\eta_t^2$ and the distribution of earnings $\exp(y_t)$ at time t has an asymptotic Pareto tail with Pareto coefficient $\alpha_t = 1/\eta_t$. Equation (6) therefore describes the dynamics of both a broad measure of earnings inequality (the variance of log earnings) and top-earnings inequality (the Pareto coefficient of the tail).

The proposition is proven by verifying that the gamma distribution given by Equation (5) solves the Kolmogorov forward equation for the process given by Equation (3). It also uses that the gamma

with $\hat{\hat{y}} = \bar{y} - \frac{\Sigma^2}{\Psi^2}$. This change of variable is equivalent to a change of currency in the measurement of earnings.

distribution features an exponential tail, and that if the distribution of log earnings features an exponential tail, then the distribution of earnings features a Pareto tail. A proof of Proposition 1 is given in Appendix Subsection A.1. For completeness, Appendix A also provides formal definitions of exponential tail and Pareto tail, relates the two, and shows that the gamma distribution indeed features an exponential tail.

Proposition 1 shows that the earnings process given by Equation (3) has three appealing theoretical features, aligning with the stylized facts, that an earnings process should satisfy if used to study the top of the earnings distribution. First, it yields an asymptotic Pareto tail in earnings. In contrast to a random growth model with reset, which generates an exact Pareto distribution (in logs an exact exponential distribution) for top earners, our earnings process generates an asymptotic Pareto distribution (since the gamma distribution in logs features an asymptotic exponential tail). This is in accordance with the broad fact that earnings are only approximately Pareto distributed at the top, not throughout the entire distribution. In the special case where $n = 1$, the earnings distribution generated by a square-root process is exactly a Pareto distribution.

Second, it also yields a Pareto tail in earnings along a transition path, not only in an eventual steady state. Contrast this with a random growth model with reset, which only features a Pareto tail in steady state. Our earnings process is thus well suited to study the transitional dynamics of top-earnings inequality.

Third, it yields an explicit and simple formula for inequality at any point in time. The dynamics given by Equation (6) describe both the evolution of the variance of log earnings and the evolution of the Pareto tail coefficient. Further, the dynamics of the Pareto coefficient as given by Equation (6) can be consistent with a quantitatively meaningful change in top-earnings inequality over a few decades. When we calibrate the earnings process in Section 3, it will become apparent that the process can quantitatively account for the dynamics of top-earnings inequality over the life cycle. By contrast, a random-growth process with reset generates counterfactually slow convergence, particularly in the tail, as shown in Gabaix et al. (2016).

3. Quantitative model evaluation

We now quantitatively evaluate whether the earnings process described above can match key empirical facts for earnings over the life cycle. In particular, we calibrate the square-root earnings process using as targeted moments the covariance structure of earnings over the life cycle, the Pareto tail coefficient over the life cycle, and a proxy for the state-dependency of earnings risk (described below), all taken from Norwegian administrative data. The covariance structure of earnings and the Pareto tail coefficient over the life cycle are both qualitatively in line with evidence from other countries than Norway, and we thus believe that the quantitative evaluation in this section is likely to carry over to other settings.[7]

3.1. Quantitative model

For our quantitative analysis, we reformulate the earnings process in discrete time and add transitory earnings shocks. Our earnings process, in discrete time and with transitory shocks, is given by the following model.

Model 1. *The* square-root model *is given by*

$$y_t = z_t + \sigma \epsilon_t, \tag{7}$$

$$z_t = \max\left\{\theta \bar{z} + (1-\theta) z_{t-1} + \Psi \sqrt{z_{t-1}} \nu_t, 0\right\}. \tag{8}$$

The max operator needs to be included since, in discrete time, there is a non-zero probability of a shock sufficiently large such that the resulting z_t would be negative if not set to zero. In practice in our quantitative exercise below, this happens very rarely.

We compare this model with the canonical model of earnings, an $AR(1)$ process for permanent earnings with transitory earnings shocks.

Model 2. *The* canonical model *of earnings dynamics is given by*

$$y_t = z_t + \sigma \epsilon_t, \tag{9}$$

$$z_t = (1-\theta) z_{t-1} + \Sigma \nu_t. \tag{10}$$

For consistency with Proposition 1, the permanent earnings shocks are normally distributed, $\nu_t \sim N(0,1)$. The theory provides no guidance for the distribution of the transitory earnings shocks, ϵ_t. We will therefore consider two cases: (i) normally distributed transitory shocks, $\epsilon_t \sim N(0,1)$, and (ii) Laplace distributed transitory shocks, $\epsilon_t \sim \text{Laplace}(0, 1/\sqrt{2})$.[8] It turns out that the Laplace distribution

[7] For the covariance structure of earnings, one can compare our calibrated parameters for the canonical model with parameters obtained in Krueger et al. (2010). The Pareto tail coefficients over the life cycle, displayed in Figure 1b, can be directly compared with the corresponding moments in Badel et al. (2018).

[8] The Laplace distribution is also known as the double exponential distribution. The density of $\text{Laplace}(0, 1/\sqrt{2})$ is given by $\frac{1}{\sqrt{2}} \exp\left(-\sqrt{2}|x|\right)$, with variance 1.

provides a better fit to the data. Further, the Laplace distribution is also consistent with the earnings growth distribution featuring "double Pareto tails" (Guvenen et al., 2021) since a Laplace distribution in log earnings growth *is* a symmetric double Pareto distribution.

3.2. Calibration

We calibrate the parameters of the square-root model and the canonical model using simulated method of moments.

Data sources The datasets used in our analysis are derived from Norwegian administrative records, covering residents from 1993 to 2017. This panel data contains annual details extracted from tax records, including various income sources. We link this dataset with the population registry, which provides demographic information such as gender and year of birth.

Our analysis focuses on pre-tax earnings, both from employment and self-employment. Our measure of earnings also includes work-related transfers, such as sick benefits and parental leave. All values are in Norwegian krone (NOK) and deflated to their 2011 real values. To make our results comparable with the rest of the literature, we select a balanced sample comprising male individuals born in Norway. We focus on prime working age individuals, aged 35 to 50, who exhibit strong attachment to the labor force throughout the relevant period.[9] With this sample selection, our data consists of 165,571 males born in Norway, spanning 8 cohorts born between 1959 and 1966. Our measure of idiosyncratic log earnings is the residual obtained from regressing log earnings on age and year dummies within our balanced sample, hence controlling for age and time effects.

Our definition of earnings includes self-employment income, which could potentially be problematic (e.g., parts of self-employment income may be better viewed as capital income). For our application, this distinction between wage income and self-employment income turns out to not be important. In Appendix B, we use only wage income, without self-employment income, as an alternative measure of earnings which we use to calibrate the square-root model and the canonical model. None of the results are affected by this change in definition of earnings.

Parameters to calibrate For the square-root model, we calibrate the three parameters of the permanent-earnings process, θ, Ψ, and $\bar{z}$, the standard deviation of the transitory shocks, σ, and the initial top-earnings inequality of permanent earnings, η_0. For the canonical model, we calibrate the two parameters of the permanent-earnings process, θ and Σ, the standard deviation of the transitory shocks, σ,

[9]We follow the definition of strong labor force attachment outlined in Crawley et al. (2022).

and, finally, we initialize the permanent-earnings distribution as a gamma distribution with parameters n and η_0. For both models, we thus calibrate 5 parameters.

Details on calibration targets We calibrate our model to three sets of targets.

First, we target the variance-covariance structure of earnings over the life cycle. This is a natural target for the canonical model, and since we want to compare our model with the canonical model we also calibrate our model to the variance-covariance structure. We compute a single empirical variance-covariance matrix for our pooled sample, based on our measure of idiosyncratic earnings.

We use the covariance moments of log earnings in levels, not first differences. It is well known that estimating the canonical model in first differences and levels yield different results (Heathcote et al., 2010). Crawley et al. (2022) find that introducing a "passing" shock, a short-lived but not fully transitory shock, reconciles the calibrated parameter values from calibrating to level moments and first-difference moments. They also show that the parameter values estimated in levels without a passing shock are consistent with the extended model, and thus that the level moments are more appropriate targets for the canonical model.

Second, we target the age-specific Pareto tail coefficient over the life cycle. Top-earnings inequality is increasing substantially over the life cycle, something which has been challenging for earnings processes to match. We calculate the age-specific Pareto coefficient above the 90th percentile by maximum likelihood, controlling for time-specific effects and adjusting the estimated age effects to align with the empirical Pareto coefficient for individuals aged 45 in 2007, following Badel et al. (2018).

Third, we target an empirical counterpart of the state-dependency of earnings risk. In particular, for each age, we run regressions of log earnings growth squared on log earnings, $(\Delta y_{it})^2 \sim a + by_{it-1}$ for above-median earners. The dependent variable, the square of earnings growth, is a computationally convenient proxy for the variance of earnings growth, which we target since we run corresponding regressions also for the simulated data during the estimation. In the data, the bottom of the earnings distribution also has very volatile earnings growth, which our parsimonious earnings process cannot capture but which an extended earnings process with transition in and out of employment likely could. Therefore, we restrict the regression to above-median earners.

A positive coefficient b cannot be directly interpreted as state-dependency of earnings risk, since the transitory shocks also contribute to this regression. Top earners are more likely to have experienced positive large transitory shocks, which means that some top earners expect substantial mean reversion in the future. Nonetheless, the coefficient b is clearly an informative moment for whether our model

Model	Shocks	Objective	θ	$\bar{z}$	Ψ	Σ	σ	Var(z_0)	z_0 top ineq.
Square root	Laplace	0.125	0.015	1.451	0.110		0.162	0.110	0.173
Square root	Normal	0.361	0.014	1.495	0.106		0.197	0.108	0.172
Canonical	Laplace	0.539	0.001			0.074	0.174	0.097	0.246
Canonical	Normal	0.751	0.001			0.070	0.215	0.098	0.233

Table 1: Estimation results for the square-root model and the canonical model, using Laplace distributed and normally distributed transitory shocks. Var(z_0) and z_0 top ineq. refers to the variance and tail inequality of the initial (gamma) distribution of permanent earnings. For the canonical model both of these are free parameters whereas for the square-root model they are subject to the restriction $n = \frac{2\theta\bar{z}}{\Psi^2}$.

generates a meaningful degree of state-dependency of earnings risk.

Details on calibration We minimize the mean of square differences between the model-generated moments and the data moments, with weights 0.15^{-2}, 1.5^{-2}, and 0.08^{-2} for the covariance moments, Pareto coefficient moments, and regression moments respectively. The weights are chosen, ad hoc, to match the scale of the plots in Figure 2.[10]

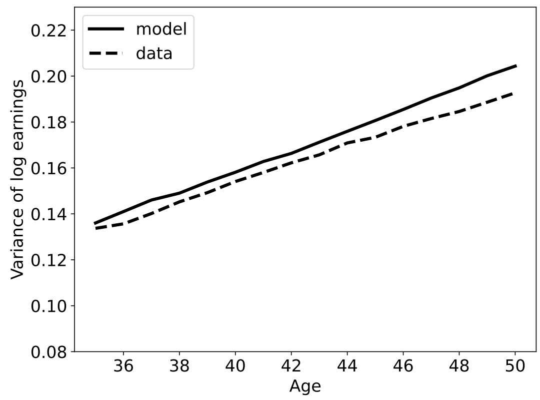
3.3. Calibrated model fit

The calibrated parameter values, and the objective function value for each calibration, are given in Table 1. The square-root model provides a better fit to the data than the canonical model, and for each model, Laplace distributed transitory shocks provide a better fit than normally distributed transitory shocks. For all earnings processes, and in line with a broad literature, permanent earnings shocks are very persistent ($1 - \theta$ is close to 1, with θ ranging from 0.001 to 0.015). The standard deviation of the transitory shocks is comparable to the literature, with σ ranging from 0.162 to 0.215. For the canonical model, the standard deviation of the permanent earnings shocks is also comparable to the literature with Σ from 0.07 to 0.074.[11]

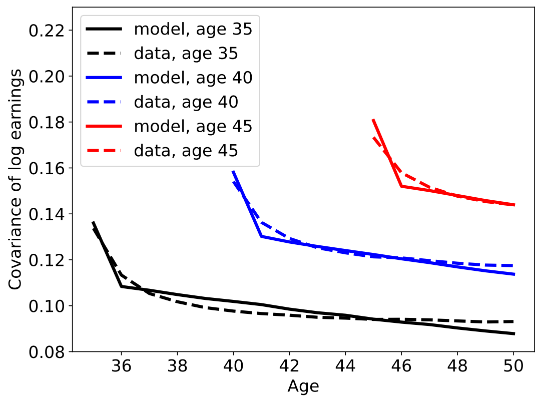
The square-root model parameters $\bar{z}$ and Ψ are not as easily compared to the previous literature. From Equation (6), the parameter Ψ controls the rate of increase in both the variance of log earnings and the tail coefficient. The parameter $\bar{z}$ controls the level of the variance of log earnings relative to the level of top-earnings inequality, through its effect on the shape parameter of the gamma distribution of permanent earnings, $n = \frac{2\theta\bar{z}}{\Psi^2}$. In particular, $n \to \infty$ ($\bar{z} \to \infty$) corresponds to the canonical model,

[10] The simulations are conducted with 100 000 simulated individuals. We perform a global search using Alisdair McKay's Python implementation of Arnoud et al. (2022)'s TikTak algorithm, with 20 000 global search values, of which the 500 with the best objective values are used as starting points for local search using the Nelder-Mead algorithm. After 30 local searches, the algorithm shrinks the starting points toward the current best parameter values.

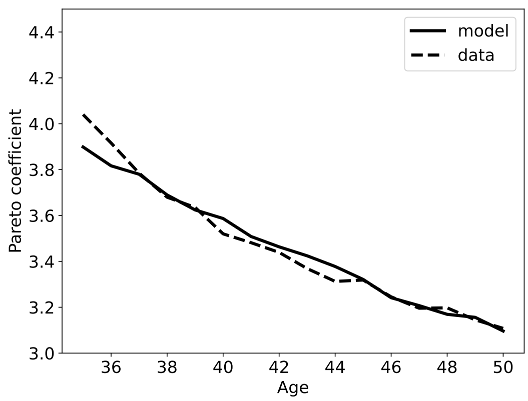
[11] For example, Crawley et al. (2022) report a standard deviation of transitory earnings shocks ranging from 0.14 to 0.18 and a standard deviation of permanent earnings shocks ranging from 0.06 to 0.07 for the canonical model with $\theta = 0$ estimated in levels on a similar but longer panel of Norwegian administrative data.



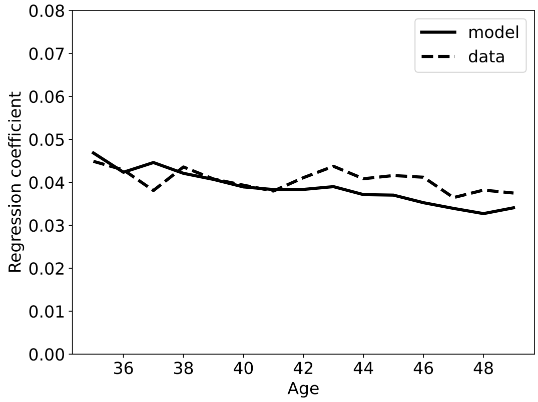
(a) The cross-sectional variance of log earnings for data and for the estimated model.



(b) The covariance of log earnings for data and for the estimated model.



(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.

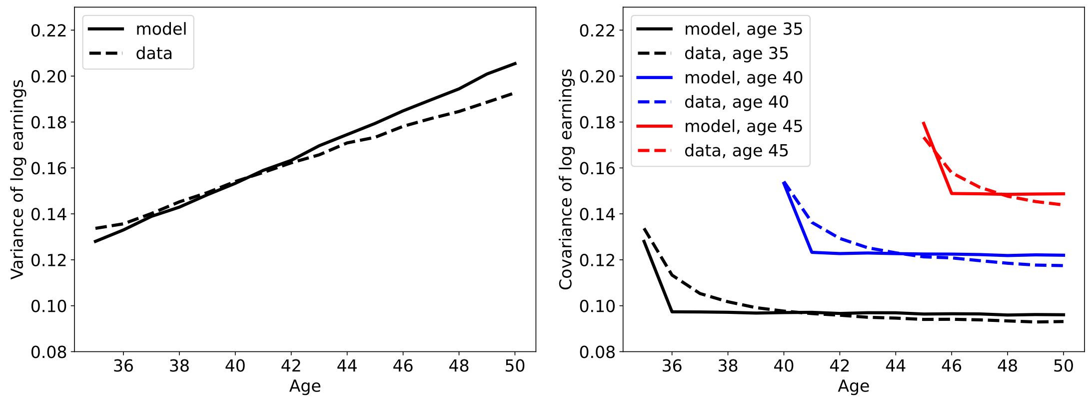


(d) The age-specific regression coefficient b for the regression $(\Delta y_{it+1})^2 \sim a + by_{it}$, for data and for the estimated model.

Figure 2: Model fit for the square-root model with Laplace distributed transitory shocks.

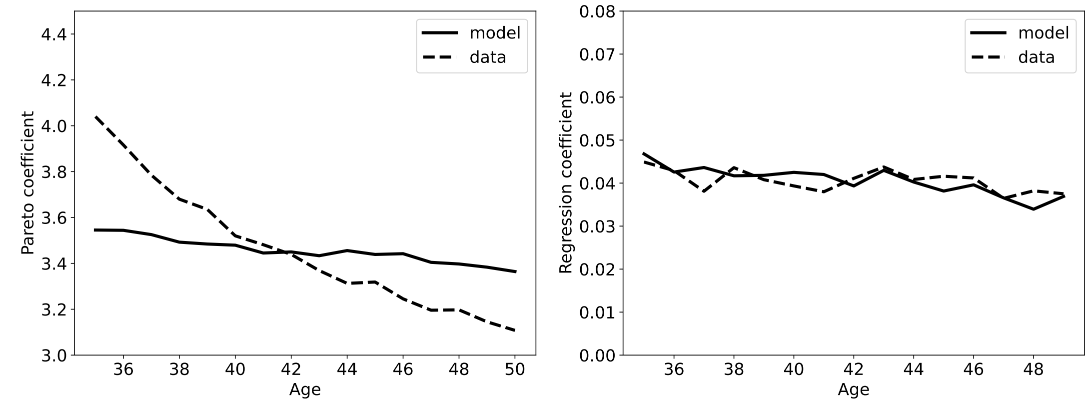
with a log-normal distribution of permanent earnings, while $n = 1$ corresponds to a Pareto distribution of permanent earnings.

The square-root model with Laplace distributed transitory shocks provides an excellent fit to the moments, as shown in Figure 2. In Figure 2a, we display the increase in the cross-sectional variance of log earnings over the life cycle. The variance of log earnings is increasing approximately linearly with age in the data, and the model matches this feature well. In Figure 2b, we display the covariance of log earnings over the life cycle. An earnings process with a permanent-transitory dichotomy generates a piecewise linear covariance structure for a given age, where the contemporaneous covariance of log earnings is higher, due to the transitory component, than the covariance of log earnings with other time periods. As is evident from Figure 2b, the calibrated model essentially fits these piecewise linear curves



(a) The cross-sectional variance of log earnings for data and for the estimated model.

(b) The covariance of log earnings for data and for the estimated model.



(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.

(d) The age-specific regression coefficient b for the regression $(\Delta y_{it+1})^2 \sim a + by_{it}$, for data and for the estimated model.

Figure 3: Model fit for the canonical model with Laplace distributed transitory shocks.

as well as possible to the data. Figure 2c shows the age-specific Pareto coefficient, the inverse of top-earnings inequality. In the data, this moment declines substantially over the life cycle as top-earnings inequality increases. The model matches the data very well, with the Pareto coefficient falling with age. Finally, Figure 2d shows the age-specific regression coefficient b for the regression $(\Delta y_{it+1})^2 \sim a+by_{it}$, for above-median earners. The coefficient b is positive and essentially constant over the life cycle. The model also matches these moments very well. The square-root model with normally distributed shocks provides a good fit to the covariance moments and to the Pareto tail coefficient over the life cycle, but it cannot match the state-dependency of earnings risk. The fit of the square-root model with normally distributed transitory shocks is shown in Appendix B, Figure B.1.

Our comparison model, the canonical model performs significantly worse. As shown in Figure 3, although the canonical model with Laplace distributed transitory shocks matches the variance moments, the covariance moments, and the age-specific regression moments, it simply cannot match the substantial increase in top-earnings inequality over the life cycle. The calibration of the canonical model does generate a Pareto tail in earnings by assuming as an initial condition that earnings are asymptotically Pareto distributed. As a result of the calibration, the average Pareto coefficient over the life cycle is broadly in line with the data. However, since the canonical model cannot generate dynamics in top-earnings inequality over the life cycle, the Pareto tail coefficient is essentially constant over time. The canonical model is thus unsuitable as an earnings process for applications where the dynamics of top-earnings inequality are central. The canonical model with normally distributed transitory shocks performs even worse, with its fit shown in Appendix B, Figure B.2.

In Appendix B, we also display corresponding results for the two models calibrated to wage income data. The results are very close to the ones shown in the main text. The square-root model with Laplace distributed transitory shocks matches the wage income moments slightly better than the earnings moments, with an objective function value of 0.108 rather than 0.125.

We therefore conclude from the calibration exercise that the square-root model matches the secular increase in top-earnings inequality over the life cycle, while the canonical model does not, in line with the theoretical results in Section 2.

Identification heuristics Here, we provide some intuition for the identification of the parameters and why the square-root model matches the moments well. The empirical variance of log earnings is increasing steadily, and approximately linearly, with age, which implies both for the square-root model and for the canonical model that permanent earnings shocks are highly persistent ($\theta \approx 0$). With $\theta \approx 0$, top-earnings inequality is also increasing with age and approximately linearly, from Equation (6), and in line with the data.

From Equation (6), the parameter Ψ controls the rate of increase in both the variance of log earnings and the tail coefficient, i.e., the slopes in Figure 2a and Figure 2c. The parameter $\bar{z}$ controls the level of the initial variance relative to the level of initial top-earnings inequality, through its effect on the shape parameter of the gamma distribution of permanent earnings, $n = \frac{2\theta\bar{z}}{\Psi^2}$, which can shift the intercept in Figure 2a keeping the intercept in Figure 2c fixed. The transitory shock standard deviation σ controls the drop in covariance relative to the contemporaneous variance seen in Figure 2b. Finally, given the above, there are no parameters left to match the regression moments, i.e., the state dependency of earnings risk. It turns out that, with Laplace distributed transitory shocks, also these moments are matched.

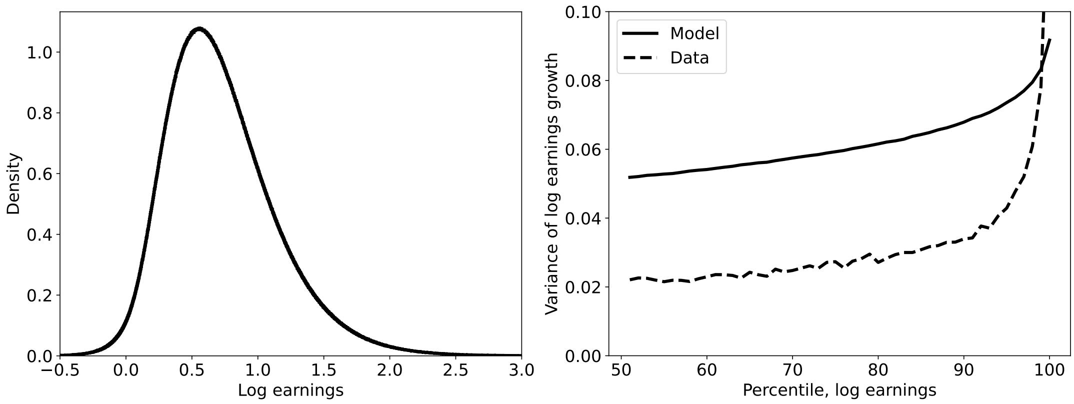
3.4. Inspecting the earnings process

The implied earnings distribution of our calibrated model satisfies a qualitative stylized fact used as motivation for the canonical model: the distribution of log earnings is “bell shaped”. In Figure 4a, we display the log earnings distribution of the estimated earnings process. The implied log earnings distribution is qualitatively “bell shaped” but with a fatter right tail than a normal distribution.

In Figure 4b, we display the variance of earnings growth for the percentiles of earners above the median. This plot corresponds to the calibration target for the state dependency of earnings risk, but rather than running a regression with squared earnings growth for each age, we plot the variance of earnings growth as a function of the rank in the earnings distribution. The variance of earnings growth is increasing with earnings, with a slope as in the data, except at the very top of the earnings distribution. However, the level of the earnings-growth variance is too high in our model relative to the data. This level difference should not be too surprising, since it is well known that the canonical model estimated in levels overestimates the variance of earnings growth relative to data.[12] Introducing a “passing” shock, as in Crawley et al. (2022), would likely shift the level of the earnings-growth variance in our model to match the data.

From the quantitative analysis, we conclude that the square-root model is able to quantitatively match the variance-covariance structure of earnings while simultaneously matching the dynamics of top-earnings inequality over the life cycle. It also generates a distribution of earnings which, qualitatively, as the canonical model, satisfies the stylized fact that the distribution of log earnings is “bell shaped”. The square-root model implies that earnings risk is state dependent, with higher risk for top earners. To ensure that the model does not generate an unreasonable degree of state dependency, we constrain the model to match an empirical proxy for the state-dependency of earnings risk, namely the positive relationship between log earnings growth squared and log earnings. The model also matches these moments well. By contrast, the canonical model can simply not match the dynamics of top-earnings inequality over the life cycle, and is thus unsuitable for applications where top-earnings inequality is central.

[12] The variance in earnings growth of the canonical model is given by $\Sigma^2 + 2\sigma^2$. Approximately, $\Sigma^2 + \sigma^2$ is equal across estimations in levels and first differences but the variance of transitory shocks, σ^2, is larger when estimated in levels (see Table 7A in Krueger et al. (2010)). As a result, the variance of earnings growth is larger for earnings processes estimated in levels than estimated in first differences. Finally, since the estimations in first differences target the variance of earnings growth, they align with the earnings growth in the data. Thus, earnings processes estimated in levels overestimate the variance of log earnings growth.



(a) The log earnings distribution for the calibrated model.

(b) The variance of log earnings growth as a function of log earnings, for above median earners, for the calibrated model.

Figure 4: The square-root model with Laplace distributed transitory shocks generates a distribution of earnings that also satisfies the stylized fact that the distribution of log earnings is "bell shaped". Variance of log earnings growth, for earners above the median, is increasing with earnings with a slope as in the data but with a difference in levels and without the very high variance at the very top of the earnings distribution.

4. Concluding remarks

We introduced a simple generalization of the canonical model, a square-root process. It retains the parsimony of the canonical model, it matches the covariance structure of earnings well, and, in contrast to the canonical model, it not only generates a Pareto tail in earnings but it also matches the dynamics of top-earnings inequality over the life cycle. The earnings process is also tractable, with only one state variable, and thus easy to use in quantitative macroeconomic models.

The square-root process implies a degree of state dependency of earnings risk. This implication of the process is consistent with the data: the process, quantitatively, matches an empirical proxy for the state dependency of earnings risk, namely the positive relationship between the variance of log earnings growth and log earnings. Therefore, the model, in a parsimonious way, generates a quantitatively realistic degree of state dependency of earnings risk.

The next steps for this research agenda is to use the square-root process to reevaluate several quantitative macroeconomic questions. In particular, the state dependency of earnings risk implies that top earners have a stronger precautionary motive with implications for the savings behavior of the rich (Fagereng et al., 2019, Straub, 2019). Evaluating the quantitative implications for wealth accumulation is a natural next step.

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Appendix

A. Mathematical appendix

In this appendix, we provide definitions of Pareto tails and exponential tails, show how they are related, and show that a Pareto tail implies regular variation. All results are completely standard and only included for completeness. In Subsection A.1, we provide the proof of Proposition 1 from the main text.

Definition A.1. *A distribution with probability density function f is said to have a Pareto tail with tail coefficient α if*

$$\lim_{x\to\infty} \frac{xf(x)}{\overline{F}(x)} = \alpha$$

where $\overline{F}(x) = 1 - F(x)$ is the countercumulative distribution function.

Definition A.2. *A distribution with probability density function f is said to have an exponential tail with tail coefficient a if*

$$\lim_{x\to\infty} \frac{f(x)}{\overline{F}(x)} = a$$

where $\overline{F}(x) = 1 - F(x)$ is the countercumulative distribution function.

Proposition A.1. *If X is distributed according to a distribution with a Pareto tail with tail coefficient α, then $Y = \log(X)$ is distributed according to a distribution with an exponential tail with tail coefficient α, and vice versa.*

Proof. The result follows from a simple change of variables. If $Y = \log(X)$, then Y has density function $g(y) = f(e^y)e^y = f(x)x$ with $x = e^y$ and countercumulative distribution function $\overline{G}(y) = \overline{F}(e^y) = \overline{F}(x)$. Thus,

$$\lim_{y\to\infty} \frac{g(y)}{\overline{G}(y)} = \lim_{x\to\infty} \frac{f(x)x}{\overline{F}(x)} = \alpha.$$

The proof of the reverse implication is analogous. □

Proposition A.2. *A gamma distribution with probability density function*

$$f(y) = \frac{(\eta)^{-n}}{\Gamma(n)} y^{n-1} \exp(-y/\eta)$$

has an exponential tail with tail coefficient $1/\eta$.

Proof. Proving the proposition amounts to showing that $\lim_{x\to\infty} \frac{x^{n-1}\exp(-x/\eta)}{\int_x^\infty y^{n-1}\exp(-y/\eta)dy} = 1/\eta$, which is true if

$$\lim_{x\to\infty} \int_x^\infty \left(\frac{y}{x}\right)^{n-1} \exp(-(y-x)/\eta)dy = \eta.$$

By a change of variable, the integral is equal to

$$\lim_{x\to\infty} \int_0^\infty \left(1+\frac{y}{x}\right)^{n-1} \exp(-y/\eta)dy = \int_0^\infty \lim_{x\to\infty} \left(1+\frac{y}{x}\right)^{n-1} \exp(-y/\eta)dy \int_0^\infty \exp(-y/\eta)dy = \eta$$

where the monotone convergence theorem justifies interchanging the integral and the limit. □

Putting the propositions together, if log earnings is gamma distributed, then log earnings has an exponential tail and earnings has a Pareto tail.

Finally, in the literature (e.g., Resnick (2007)), the following definition of regularly varying functions is standard:

Definition A.3. *A measurable function* $U : \mathbb{R}_+ \to \mathbb{R}_+$ *is regularly varying at* ∞ *with index* ρ *if for all* $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\rho.$$

The notion of regularly varying functions generalizes our definition of a Pareto tail. Indeed, our definition of a Pareto tail implies regular variation.

Proposition A.3. *If a distribution with countercumulative distribution function* $\overline{F}$ *has a Pareto tail, then* $\overline{F}$ *is regularly varying at* ∞ *with index* $-\alpha$.

Proof. Note that

$$\lim_{t\to\infty} \log \overline{F}(tx) - \log \overline{F}(t) = -\lim_{t\to\infty} \int_t^{tx} \frac{f(y)}{\overline{F}(y)}dy = -\lim_{t\to\infty} \int_t^{tx} \frac{f(y)y}{\overline{F}(y)}\frac{1}{y}dy = -\alpha \log x$$

where the last step follows from $\lim_{t\to\infty} \frac{f(y)y}{\overline{F}(y)} = \alpha$. Thus, $\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-\alpha}$. □

A.1. Proof of Proposition 1

Here, we prove Proposition 1 from the main text, restated here for convenience.

Proposition 1. *Let log earnings y_t follow the square-root process given by*

$$dy_t = \theta\left(\bar{y} - y_t\right)dt + \Psi\sqrt{y_t}dW_t. \tag{3}$$

Assume that the initial distribution of log earnings is given by the gamma distribution,

$$f_0(y) = \frac{(\eta_0)^{-n}}{\Gamma(n)} y^{n-1}\exp(-y/\eta_0) \tag{4}$$

with $n = \frac{2\theta\bar{y}}{\Psi^2}$. Then the cross-sectional distribution of log earnings at a given point in time is also distributed according to a gamma distribution,

$$f_t(y) = \frac{(\eta_t)^{-n}}{\Gamma(n)} y^{n-1}\exp(-y/\eta_t) \tag{5}$$

where η_t evolves according to

$$d\eta_t = (\Psi^2/2 - \theta\eta_t)dt. \tag{6}$$

The distribution of log earnings at time t has variance given by $n\eta_t^2$ and the distribution of earnings $\exp(y_t)$ at time t has an asymptotic Pareto tail with Pareto coefficient $\alpha_t = 1/\eta_t$. Equation (6) therefore describes the dynamics of both a broad measure of earnings inequality (the variance of log earnings) and top-earnings inequality (the Pareto coefficient of the tail).

Proof. The proposition is proven by guessing and verifying that the gamma distribution solves the Kolmogorov forward equation. For the process given by Equation (3), the Kolmogorov forward equation is given by

$$\begin{aligned}\frac{df_t}{dt} =& -\theta\left(\frac{n\Psi^2}{2\theta} - y_t\right)\frac{df_t}{dy} + \theta f_t + \frac{1}{2}\Psi^2\left(y_t\frac{d^2 f_t}{dy^2} + 2\frac{df_t}{dy}\right)\\ =& \theta f_t - ((n+1)\Psi^2 - \theta y_t)\frac{df_t}{dy} + \frac{1}{2}\Psi^2 y_t\frac{d^2 f_t}{dy^2}.\end{aligned}$$

We guess that the gamma distribution, as given by Equation (5), solves the Kolmogorov forward equation with the dynamics given by Equation (6). Plugging in the derivatives of Equation (5) yields

$$\begin{aligned}\frac{df_t}{dt}\frac{1}{f_t} =& \theta - ((n/2-1)\Psi^2 - \theta y_t)\left(\frac{n-1}{y} - \frac{1}{\eta_t}\right) + \frac{1}{2}\Psi^2\left(y_t\left(-\frac{n-1}{y^2} + \left(\frac{n-1}{y} - \frac{1}{\eta_t}\right)^2\right)\right)\\ =& \left(-\frac{n}{\eta_t} + \frac{y}{\eta_t^2}\right)\left(\frac{\Psi^2}{2} - \theta\eta_t\right).\end{aligned}$$

To conclude the proof, we differentiate Equation (5) with respect to η_t,

$$\frac{df_t}{d\eta_t}\frac{1}{f_t} = -\frac{n}{\eta_t} + \frac{y}{\eta_t^2}.$$

We therefore obtain

$$\frac{df_t}{dt} = \frac{df_t}{d\eta_t}\frac{d\eta_t}{dt}$$

when $\frac{d\eta_t}{dt} = \frac{\Psi^2}{2} - \theta\eta_t$.

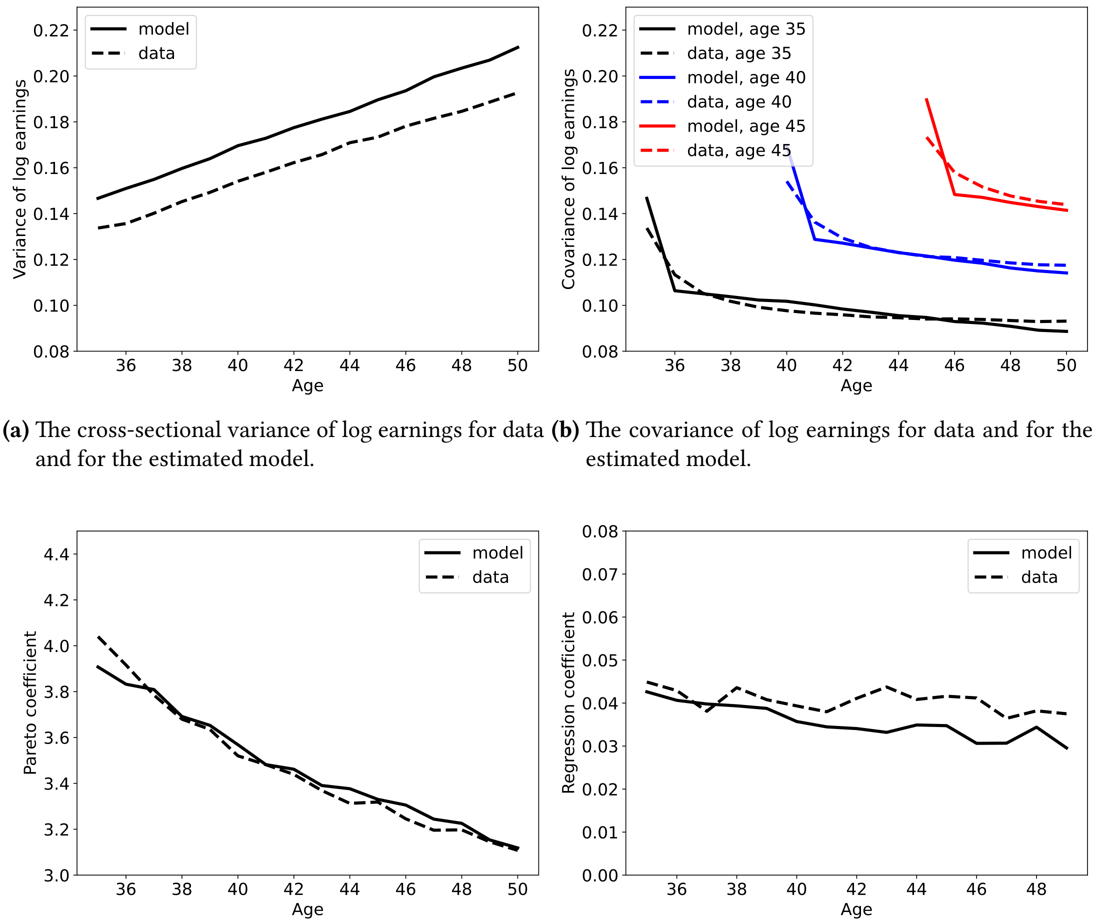
It is therefore clear that Equation (5) together with Equation (6) solve the Kolmogorov forward equation for the process given by Equation (3).

Finally, the distribution of permanent earnings features a Pareto tail if log earnings is gamma distributed, as given by the propositions in this section. □

B. Additional calibration results

B.1. Calibration results with normally distributed transitory shocks

Here, we provide some additional figures showing the fit of the alternative models with normally distributed transitory shocks. Figure B.1 shows the fit of the square-root model with normally distributed transitory shocks. Figure B.2 shows the fit of the canonical model with normally distributed transitory shocks.



(a) The cross-sectional variance of log earnings for data and for the estimated model.

(b) The covariance of log earnings for data and for the estimated model.

(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.

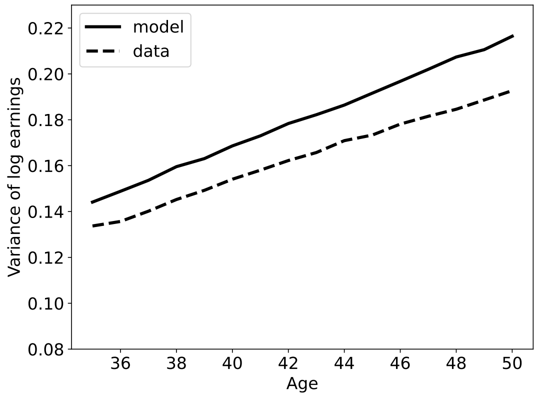
(d) The age-specific regression coefficient b for the regression $(\Delta y_{it+1})^2 \sim a + by_{it}$, for data and for the estimated model.

Figure B.1: Model fit for the square-root model with normally distributed transitory shocks.

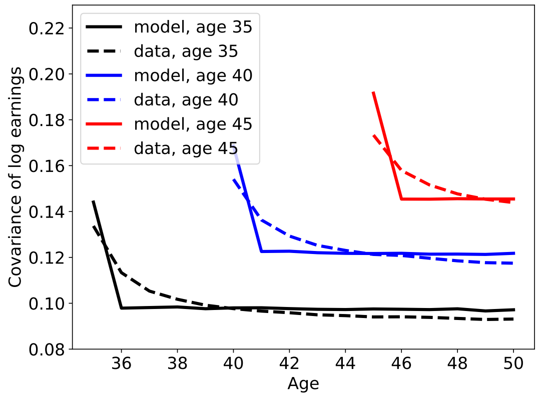
B.2. Calibration results with wage income as an alternative measure of earnings

Here, we provide some additional figures showing the fit of the models calibrated with wage income as an alternative measure of earnings. Table B.1 shows the estimation results with wage income as the measure of earnings. Figure B.3 shows the fit of the square-root model with Laplace distributed transitory shocks and Figure B.4 shows the fit of the canonical model with Laplace distributed transitory shocks.

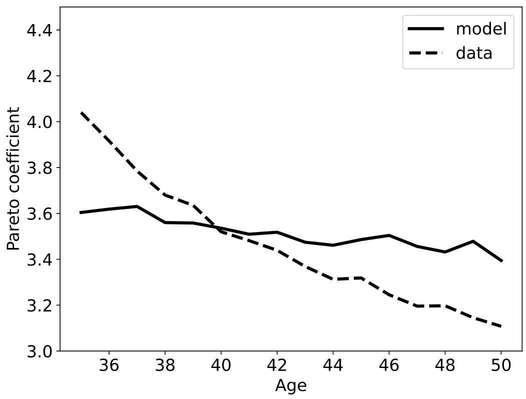
Alternative definition of earnings, wage income Wage income encompasses earnings from employment, including salaries and wages, taxable fringe payments, and sickness and parental benefits.



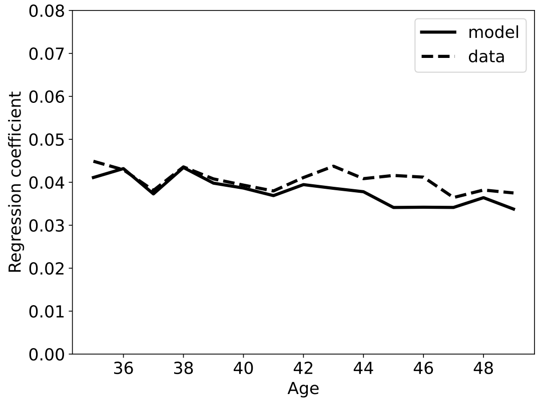
(a) The cross-sectional variance of log earnings for data and for the estimated model.



(b) The covariance of log earnings for data and for the estimated model.



(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.

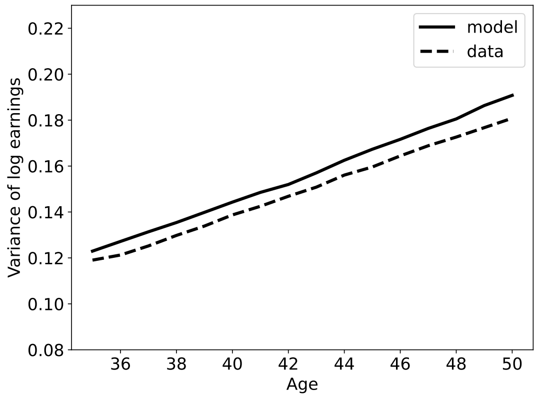


(d) The age-specific regression coefficient b for the regression $(\Delta y_{it+1})^2 \sim a + by_{it}$, for data and for the estimated model.

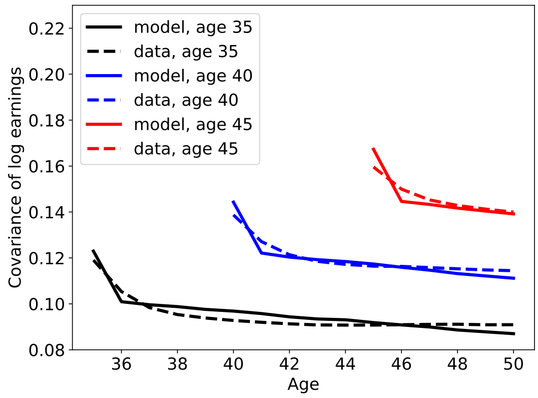
Figure B.2: Model fit for the canonical model with normally distributed transitory shocks.

Model	Shocks	Objective	θ	$\bar{z}$	Ψ	Σ	σ	Var(z_0)	z_0 top ineq.
Square root	Laplace	0.108	0.011	1.713	0.105		0.144	0.102	0.176
Square root	Normal	0.268	0.010	1.781	0.106		0.167	0.100	0.180
Canonical	Laplace	0.459	0.002			0.073	0.155	0.094	0.242
Canonical	Normal	0.650	0.000			0.070	0.196	0.092	0.255

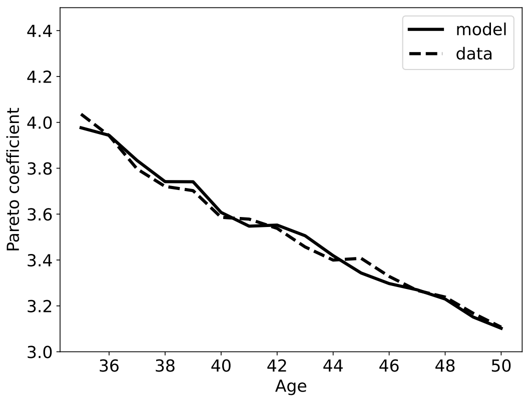
Table B.1: Estimation results with wage income as our measure of earnings for the square-root model and the canonical model, using Laplace distributed and normally distributed transitory shocks. Var(z_0) and z_0 top ineq. refers to the variance and tail inequality of the initial gamma distribution of log permanent earnings. For the canonical model both of these are free parameters whereas for the square-root model they are subject to the restriction $n = \frac{2\theta\bar{z}}{\Psi^2}$.



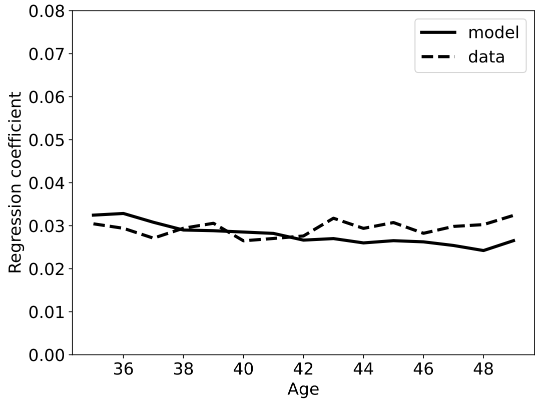
(a) The cross-sectional variance of log earnings for data and for the estimated model.



(b) The covariance of log earnings for data and for the estimated model.

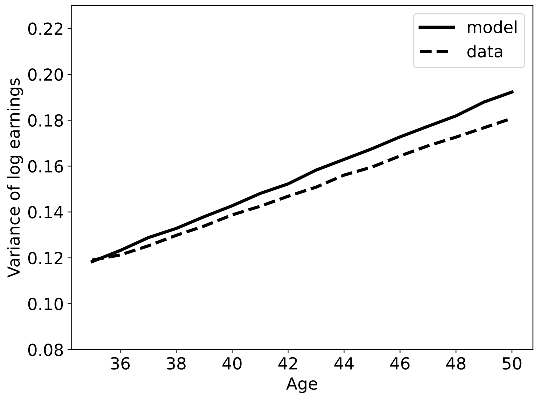


(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.

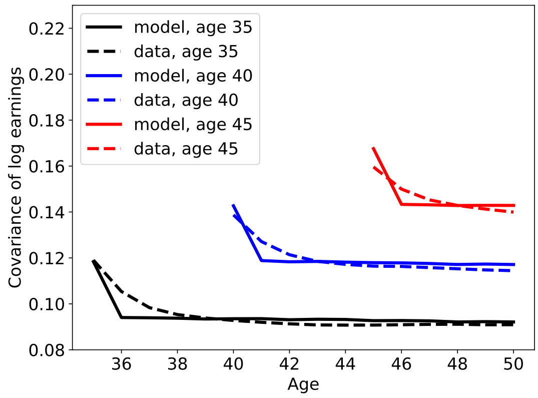


(d) The age-specific regression coefficient b for the regression $(\Delta y_{it+1})^2 \sim a + by_{it}$, for data and for the estimated model.

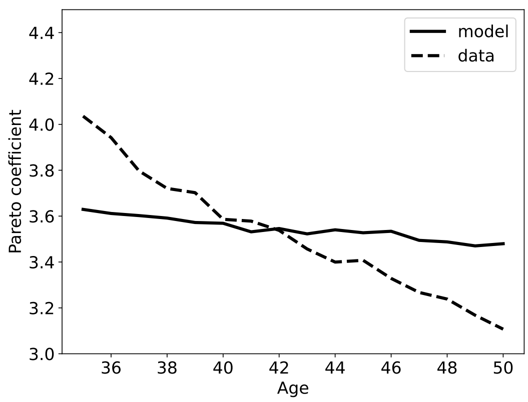
Figure B.3: Model fit with wage income as our measure of earnings for the square-root model with Laplace distributed transitory shocks.



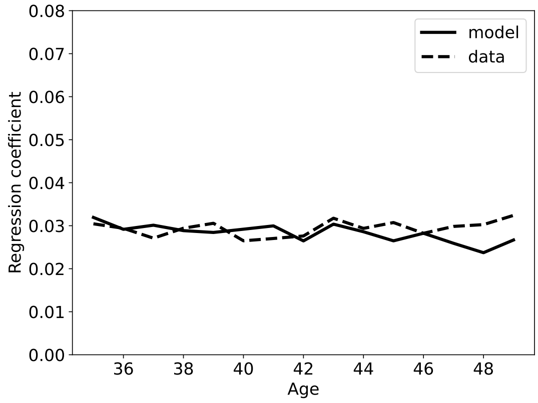
(a) The cross-sectional variance of log earnings for data and for the estimated model.



(b) The covariance of log earnings for data and for the estimated model.



(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.



(d) The age-specific regression coefficient b for the regression $(\Delta y_{it+1})^2 \sim a + by_{it}$, for data and for the estimated model.

Figure B.4: Model fit with wage income as our measure of earnings for the canonical model with Laplace distributed transitory shocks.